Calculating the Probability of an Event (The Event-Composition Method)

The following steps are used in the event-composition method to find the probability of an event:

1. Define the experiment.
2. Visualize the nature of the sample points. Identify a few to clarify your thinking.
3. Write an equation expressing the event of interest, say \( A \), as a composition of two or more events, using unions, intersections, and/or compliments. (Notice that this equates point sets.) Make certain that event \( A \) and the event implied by the composition represent the same set of sample points.
4. Apply the additive and multiplicative laws of probability to the compositions obtained in step 3 to find \( P(A) \).

**Example 1.** Of the voters in a city, 40% are Republicans and 60% are Democrats. Among the Republicans 70% are in favor of a bond issue, whereas 80% of Democrats favor the issue. If a voter is selected at random in the city, what is the probability that he or she will favor the bond issue?

**Solution.** Define the following events:

- \( F \) : Favor the bond issue.
- \( R \) : A Republican is selected.
- \( D \) : A Democrat is selected.

Then \( P(R) = 0.4 \), \( P(D) = 0.6 \), \( P(F|R) = 0.7 \), and \( P(F|D) = 0.8 \). Note from the figure below that \( F = (F \cap R) \cup (F \cap D) \). Then
\[ P(F) = P[(F \cap R) \cup (F \cap D)] = P(F \cap R) + P(F \cap D) \]

because \( F \cap R \) and \( F \cap D \) are mutually exclusive events. By the multiplicative law of probability, it follows that

\[
\begin{align*}
P(F \cap R) &= P(R) P(F|R) = (0.4)(0.7) = 0.28 \\
P(F \cap D) &= P(D) P(F|D) = (0.6)(0.8) = 0.48.
\end{align*}
\]

Therefore,

\[ P(F) = 0.28 + 0.48 = 0.76. \quad \blacksquare \]

**Example 2.** The birthdays of 20 randomly selected persons were recorded. The probability is 0.5886 that each person has a different birthday. What is the probability that at least one pair of individuals share a birthday?

**Solution.** Define the events \( A \) and \( B \) as follows:

\[
A : \quad \text{Each person has a different birthday.} \\
B : \quad \text{At least one pair of individuals share a birthday.}
\]

Then event \( B \) is the set of all sample points in the sample space \( S \) that are not in \( A \), that is, \( B = \overline{A} \). Therefore, by the complementary law of probability, we have

\[ P(B) = 1 - P(A) = 1 - 0.5886 = 0.4114. \quad \blacksquare\]

**Example 3.** Two applicants are randomly selected from among five who have applied for a job. Find the probability that exactly one of the two best applicants is selected.

**Solution.** Define the following events:

\[
\begin{align*}
A & : \quad \text{Exactly one of the two best applicants is selected.} \\
B & : \quad \text{Draw the best and one of the three poorest applicants.} \\
C & : \quad \text{Draw the second best and one of the three poorest applicants.}
\end{align*}
\]

Clearly, events \( B \) and \( C \) are mutually exclusive and \( A = B \cup C \). Now let \( D_1 = B_1 \cap B_2 \), where

\[
\begin{align*}
B_1 & : \quad \text{Draw the best applicant on the first draw,} \\
B_2 & : \quad \text{Draw one of the three poorest applicants on the second draw,}
\end{align*}
\]

and \( D_2 = B_3 \cap B_4 \), where

\[
\begin{align*}
B_3 & : \quad \text{Draw one of the three poorest applicants on the first draw,} \\
B_4 & : \quad \text{Draw the best applicant on the second draw.}
\end{align*}
\]

Note that \( B = (B_1 \cap B_2) \cup (B_3 \cap B_4) = D_1 \cup D_2 \).
Similarly, let \( G_1 = C_1 \cap C_2 \), where

\[
C_1 : \text{Draw the second best applicant on the first draw},
\]
\[
C_2 : \text{Draw one of the three poorest applicants on the second draw},
\]

and \( G_2 = C_3 \cap C_4 \), where

\[
C_3 : \text{Draw one of the three poorest applicants on the first draw},
\]
\[
C_4 : \text{Draw the second best applicant on the second draw}.
\]

Note that \( C = (C_1 \cap C_2) \cup (C_3 \cap C_4) = G_1 \cup G_2 \).

Note also that \( D_1 \) and \( D_2 \) and \( G_1 \) and \( G_2 \) are pairs of mutually exclusive events and that

\[
A = B_1 \cup C = (D_1 \cup D_2) \cup (G_1 \cup G_2) = D_1 \cup D_2 \cup G_1 \cup G_2
\]
\[
= (B_1 \cap B_2) \cup (B_3 \cap B_4) \cup (C_1 \cap C_2) \cup (C_3 \cap C_4).
\]

Applying the multiplicative law of probability, we have

\[
P(B_1 \cap B_2) = P(B_1) \cdot P(B_2 | B_1) = \frac{1}{5} \cdot \frac{3}{4} = \frac{3}{20},
\]
\[
P(B_3 \cap B_4) = P(B_3) \cdot P(B_4 | B_3) = \frac{3}{5} \cdot \frac{1}{4} = \frac{3}{20},
\]
\[
P(C_1 \cap C_2) = P(C_1) \cdot P(C_2 | C_1) = \frac{1}{5} \cdot \frac{3}{4} = \frac{3}{20},
\]
\[
P(C_3 \cap C_4) = P(C_3) \cdot P(C_4 | C_3) = \frac{3}{5} \cdot \frac{1}{4} = \frac{3}{20}.
\]

Therefore,

\[
P(A) = P(B_1 \cap B_2) + P(B_3 \cap B_4) + P(C_1 \cap C_2) + P(C_3 \cap C_4)
\]
\[
= \frac{3}{20} + \frac{3}{20} + \frac{3}{20} + \frac{3}{20} = \frac{12}{20} = \frac{3}{5}.
\]

**Example 4.** It is known that a patient with a disease will respond to treatment with probability equal to 0.9. If three patients with the disease are treated and respond independently, find the probability that at least one will respond.

**Solution.** Define the following events:

\( A \) : At least one of the three patients will respond.
\( B_1 \) : The first patients will not respond.
\( B_2 \) : The second patients will not respond.
\( B_3 \) : The third patients will not respond.

Observe that \( \overline{A} = B_1 \cap B_2 \cap B_3 \). It follows from the complementary law of probability that

\[
P(A) = 1 - P(\overline{A}) = 1 - P(B_1 \cap B_2 \cap B_3).
\]
Repeatedly applying the multiplicative law of probability, we have

\[ P(B_1 \cap B_2 \cap B_3) = P(B_1 \cap B_2) P(B_3 \mid B_1 \cap B_2) \]
\[ = P(B_1) P(B_2 \mid B_1) P(B_3 \mid B_1 \cap B_2), \]

where, because the events are independent,

\[ P(B_2 \mid B_1) = P(B_2) = 0.1 \quad \text{and} \quad P(B_3 \mid B_1 \cap B_2) = P(B_3) = 0.1. \]

Also, \( P(B_1) = 0.1 \). Therefore,

\[ P(A) = 1 - (0.1)(0.1)(0.1) = 0.999. \]

**Example 5.** Observation of a waiting line at a medical clinic indicates the probability that a new arrival will be an emergency case is \( p = 1/6 \). Find the probability that the \( r \)th patient is the first emergency case. (Assume that conditions of arriving patients represent independent events.)

**Solution.** The experiment consists of watching patient arrivals until the first emergency case appears. Then the sample points for the experiment are

\[ E_i : \text{The } i\text{th patient is the first emergency case, for } i = 1, 2, \ldots. \]

Because only one sample point falls in the event of interest,

\[ P(\text{rth patient is the first emergency case}) = P(E_r). \]

Now denote by \( A_i \) the event that the \( i \)th arrival is not an emergency case. Then \( E_r \) can be expressed as the intersection as follows:

\[ E_r = A_1 \cap A_2 \cap A_3 \cap \cdots \cap A_{r-1} \cap \overline{A}_r. \]

As shown in Example 4, if we repeatedly applying the multiplicative law of probability, we have

\[ P(E_r) = P(A_1) P(A_2 \mid A_1) P(A_3 \mid A_1 \cap A_2) P(A_3 \mid A_1 \cap A_2 \cap A_3) \]
\[ \cdots P(\overline{A}_r \mid A_1 \cap A_2 \cap A_3 \cdots \cap A_{r-1}). \]

Because the events \( A_1, A_2, A_3, \cdots, A_{r-1}, \) and \( \overline{A}_r \) are independent, it follows that

\[ P(E_r) = P(A_1) P(A_2) P(A_3) \cdots P(\overline{A}_r) \]
\[ = \left( \frac{5}{6} \right)^{r-1} \left( \frac{1}{6} \right) \quad \text{for } r = 1, 2, 3, \ldots. \]
Note that
\[
P(S) = P(E_1) + P(E_2) + P(E_3) + \cdots + P(E_i) + \cdots
\]
\[
= \frac{1}{6} + \left( \frac{5}{6} \right) \left( \frac{1}{6} \right) + \left( \frac{5}{6} \right)^2 \left( \frac{1}{6} \right) + \cdots + \left( \frac{5}{6} \right)^{i-1} \left( \frac{1}{6} \right) + \cdots
\]
\[
= \frac{1}{6} \sum_{i=0}^{\infty} \left( \frac{5}{6} \right)^i = \frac{\frac{1}{6}}{1 - \frac{5}{6}} = 1.
\]
This result follows from the formula for the sum of a geometric series \((a + ar + ar^2 + \cdots + ar^{i-1} + \cdots = \frac{a}{1-r} \text{ if } |r| < 1)\).

**Example 6.** A monkey is to demonstrate that she recognizes colors by tossing one red, one black, and one white ball into boxes of the same respective colors, one ball to a box. If the monkey has not learned the colors and merely tosses one ball into each box at random, find the probabilities of the following results:

(a) There are no color matches.

(b) There is exactly one color match.

**Solution.** Define the following events:

\[A_1: \text{A color match occurs in the red box.}\]
\[A_2: \text{A color match occurs in the black box.}\]
\[A_3: \text{A color match occurs in the white box.}\]

There are \(3! = 6\) equally likely ways of randomly tossing the balls into the boxes with one ball in each box. Also, there are only \(2! = 2\) ways of tossing the balls into the boxes if one particular box is required to have a color match. Hence

\[P(A_1) = P(A_2) = P(A_3) = \frac{2}{6} = \frac{1}{3}.
\]

Similarly, it is easy to see that

\[P(A_1 \cap A_2) = P(A_1 \cap A_3) = P(A_2 \cap A_3) = P(A_1 \cap A_2 \cap A_3) = \frac{1}{6}.
\]

(a) Note that

\[P(\text{no color matches}) = 1 - P(\text{at least one color match})
\]
\[= 1 - P(A_1 \cup A_2 \cup A_3)
\]
\[= 1 - [P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)]
\]
\[= 1 - \left[ 3 \left( \frac{1}{3} \right) - 3 \left( \frac{1}{6} \right) + \frac{1}{6} \right] = \frac{2}{6} = \frac{1}{3}.
\]
(b) From the graphical demonstration below, it is easy to see that

\[
P \text{(exactly one match)} = P(A_1) + P(A_2) + P(A_3) \\
-2 [P(A_1 \cap A_2) + P(A_1 \cap A_3) + P(A_2 \cap A_3)] \\
+3P(A_1 \cap A_2 \cap A_3) \\
= 3 \left( \frac{1}{3} \right) - 2 \left( 3 \frac{1}{6} \right) + 3 \left( \frac{1}{6} \right) = \frac{3}{6} = \frac{1}{2}.
\]

The three shaded regions of the following figure depicts the event that there is exactly one color match.

(Exactly one color match: shaded area)

where the region representing only the red color match is depicted as:
the region representing only the black color match is depicted as:

\[
= A_2 - A_1 \cap A_2 - A_2 \cap A_3 + A_1 \cap A_2 \cap A_3
\]

and the region representing only the white color match is depicted as:

\[
= A_3 - A_1 \cap A_3 - A_2 \cap A_3 + A_1 \cap A_2 \cap A_3
\]