Differential Equations Assignment 5

1. Show that $x = 0$ is an ordinary point of the differential equation

$$
\frac{d^2 y}{dx^2} - 2x \frac{dy}{dx} + \lambda y = 0.
$$

Next, find two linearly independent solutions of this equation and show that, if the parameter $\lambda$ is a positive integer, one of the solutions becomes a polynomial.

2. Show that $x = 0$ is a regular singular point of

$$
x \frac{d^2 y}{dx^2} + (1 - x) \frac{dy}{dx} + \lambda y = 0.
$$

Find two linearly independent solutions of this equation.

3. Show that $x = 0$ is a regular singular point of

$$
x^2 \frac{d^2 y}{dx^2} + (x^2 - x) \frac{dy}{dx} + y = 0.
$$

Find two linearly independent solutions to this equation. Note: The indicial equation has a repeated zero.

4. Show that $x = 0$ is a regular singular point of

$$
x \frac{d^2 y}{dx^2} + y = 0.
$$

Find two linearly independent solutions of this equation. Note: The indicial equation has zeros differing by a positive integer.

5. In this problem we consider Bessel’s equation of order 1:

$$
x^2 y'' + xy' + (x^2 - 1)y = 0.
$$

(a) Find one series solution for this equation. You should obtain

$$
y(x) = C x \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{2^{2n}(n+1)!n!};
$$

with the choice $C = 1/2$ this gives the standard form of the Bessel function $J_1(x)$,

$$
J_1(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(n+1)!n!} \left( \frac{x}{2} \right)^{2n+1}.
$$
(b) In order to find a second solution of Bessel’s equation of order 1, substitute
\[ y(x) = J_1(x) \ln x + \frac{1}{x} \left[ \sum_{n=0}^{\infty} b_n x^n \right], \]
to show that
\[ -b_1 + b_0 x + \sum_{n=2}^{\infty} \left[ (n^2 - 1) b_{n+1} + b_{n-1} \right] x^n = -2 \sum_{k=0}^{\infty} \frac{(-1)^k (2k + 1)}{(k+1)!k!} \left( \frac{x}{2} \right)^{2k+1}. \]

(c) Hence show that \( b_0 = -1, b_1 = 0 \), and that \( b_n \) obeys the recurrence relation
\[
(n^2 - 1) b_{n+1} + b_{n-1} = 0
\]
if \( n \) is even and, for \( k = 1, 2, 3, \ldots \),
\[
\left[ (2k + 1)^2 - 1 \right] b_{2(k+1)} + b_{2k} = \frac{(-1)^{k+1} (2k + 1)}{2^{2k}(k+1)!k!}.
\]
Deduce that \( b_j = 0 \) for all odd values of \( j \).

(d) Denoting by \( H_n \) the sum \( H_n = \sum_{k=1}^{n} \frac{1}{k} \), verify that
\[
b_{2k} = \frac{(-1)^k (H_k + H_{k-1})}{2^{2k}k!(k-1)!}
\]
and hence write down a second solution to Bessel’s equation.